

An Explicit Integration of a Problem of Motion of a Generalized Kovalevskaya Top¹

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Kovalevskaya's solution [1] of the problem of motion of a heavy rigid body about a fixed point was generalized to the case of a double constant field in [2, 3]. The corresponding Hamiltonian system has three degrees of freedom. The invariant four-dimensional submanifolds of the phase space were found in [2, 4]. The case of [2] was studied in [5]. In this paper, we consider the case of [4].

Consider a rigid body having a fixed point and satisfying the Kovalevskaya condition for the principal moments of inertia at the fixed point $\mathbf{I} = \text{diag}\{2, 2, 1\}$. Suppose that the body is placed in a force field with the potential

$$U = -(\mathbf{e}_1, \boldsymbol{\alpha}) - (\mathbf{e}_2, \boldsymbol{\beta}), \quad (1)$$

where \mathbf{e}_1 and \mathbf{e}_2 are the orthonormal vectors directed along the principal axes of inertia in the equatorial plane and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are fixed vectors in the inertial space.

Theorem 1. *Without loss of generality we can assume the vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}$ to be mutually orthogonal.*

Proof. Note that the potential (1) is invariant with respect to the substitution

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} \mapsto \Theta \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} \mapsto \Theta \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix}, \quad \Theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad \text{tg } 2\theta = \frac{2\boldsymbol{\alpha} \cdot \boldsymbol{\beta}}{\boldsymbol{\alpha}^2 - \boldsymbol{\beta}^2},$$

which leaves the pair $\mathbf{e}_1, \mathbf{e}_2$ orthonormal in the equatorial plane of the body but makes the vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}$ orthogonal to each other.

Let $\boldsymbol{\alpha}^2 = a^2$ and $\boldsymbol{\beta}^2 = b^2$. We consider the general case $a > b$ and denote $p^2 = a^2 + b^2$, $r^2 = a^2 - b^2$.

The corresponding Euler–Poisson equations are Liouville–Arnold integrable due to the first integrals [2, 3]

$$\begin{aligned} H &= \omega_1^2 + \omega_2^2 + \frac{1}{2}\omega_3^2 - (\alpha_1 + \beta_2), \\ K &= (\omega_1^2 - \omega_2^2 + \alpha_1 - \beta_2)^2 + (2\omega_1\omega_2 + \alpha_2 + \beta_1)^2, \\ G &= \frac{1}{4}(\omega_\alpha^2 + \omega_\beta^2) + \frac{1}{2}\omega_3\omega_\gamma - b^2\alpha_1 - a^2\beta_2, \end{aligned}$$

where

$$\begin{aligned} \omega_\alpha &= 2\omega_1\alpha_1 + 2\omega_2\alpha_2 + \omega_3\alpha_3, \quad \omega_\beta = 2\omega_1\beta_1 + 2\omega_2\beta_2 + \omega_3\beta_3, \\ \omega_\gamma &= 2\omega_1(\alpha_2\beta_3 - \alpha_3\beta_2) + 2\omega_2(\alpha_3\beta_1 - \alpha_1\beta_3) + \omega_3(\alpha_1\beta_2 - \alpha_2\beta_1). \end{aligned}$$

Let us introduce the functions on the phase space

$$\begin{aligned} F &= (2G - p^2H)^2 - r^4K, \quad M = (2G - p^2H)/r^4, \\ L &= \sqrt{2p^2M^2 + 2HM + 1}. \end{aligned}$$

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They are also first integrals of motion. Denote by N the set of critical points of the function F on the level $F = 0$.

Now we consider the new variables ($i^2 = -1$):

$$\begin{aligned} x_1 &= (\alpha_1 - \beta_2) + i(\alpha_2 + \beta_1), & x_2 &= \overline{x_1}, \\ y_1 &= (\alpha_1 + \beta_2) + i(\alpha_2 - \beta_1), & y_2 &= \overline{y_1}, \\ z_1 &= \alpha_3 + i\beta_3, & z_2 &= \overline{z_1}, \\ w_1 &= \omega_1 + i\omega_2, & w_2 &= \overline{w_1}, & w_3 &= \omega_3. \end{aligned} \quad (2)$$

Theorem 2. *In the domain $x_1x_2 \neq 0$ the set N is determined by the two independent equations*

$$F_1 = 0, \quad F_2 = 0, \quad (3)$$

where

$$F_1 = \sqrt{x_1x_2}w_3 - \frac{1}{\sqrt{x_1x_2}}(x_2z_1w_1 + x_1z_2w_2), \quad F_2 = \frac{i}{2}\left[\frac{x_2}{x_1}(w_1^2 + x_1) - \frac{x_1}{x_2}(w_2^2 + x_2)\right].$$

In particular, in this domain, N is a smooth four-dimensional manifold. The induced vector field on N is Hamiltonian everywhere except at the points where $L = 0$.

Proof. The proof of the first statement of the theorem is by direct calculation. The second statement follows from the invariance of the set of critical points of the first integral and from the relation $\{F_2, F_1\} = r^2L$ for the Poisson bracket.

Equations (3) were obtained in [4], but they do not describe the invariant set N as a whole because of the presence of an obvious singularity. The theorem stated above implies that N is determined globally, and, on this set, we have a completely integrable Hamiltonian system with two degrees of freedom such that the set of points where the symplectic structure degenerates is a thin set.

It is convenient to take the functions M and L as an involutive pair of first integrals.

Theorem 3. *The change of variables*

$$s_1 = \frac{x_1x_2 + z_1z_2 + r^2}{2\sqrt{x_1x_2}}, \quad s_2 = \frac{x_1x_2 + z_1z_2 - r^2}{2\sqrt{x_1x_2}} \quad (4)$$

reduces the equations of motion of the Kovalevskaya top in a double force field on the integral manifold

$$J_{m,\ell} = \{M = m, L = \ell\} \subset N$$

to the system

$$\frac{ds_1}{dt} = \frac{1}{2}\sqrt{(a^2 - s_1^2)\Phi(s_1)}, \quad \frac{ds_2}{dt} = \frac{1}{2}\sqrt{(b^2 - s_2^2)\Phi(s_2)}, \quad (5)$$

where $\Phi(s) = 4ms^2 - 4\ell s + (\ell^2 - 1)/m$. The solutions of this system can be written explicitly in terms of elliptic functions.

Proof. Let us eliminate the variables w_i ($i = 1, 2, 3$) from relations (3) and the integral equation $M = m$. The equation $L = \ell$ takes the form

$$m(x_1x_2 + z_1z_2) - \ell\sqrt{x_1x_2} + \sqrt{m^2r^4 - mr^2(x_1 + x_2) + x_1x_2} = 0. \quad (6)$$

The vectors α, β have constant length and are mutually orthogonal. In variables (2) this fact is written as

$$z_1^2 + x_1y_2 = r^2, \quad z_2^2 + x_2y_1 = r^2, \quad x_1x_2 + y_1y_2 + 2z_1z_2 = 2p^2. \quad (7)$$

Then calculating the time derivatives of variables (4) and taking into account equation (6), we obtain system (5).

Note that, by virtue of (7), the variables s_1 and s_2 satisfy the natural constraints $s_1^2 \geq a^2, s_2^2 \leq b^2$. Therefore, the real solutions of (5) oscillate in the intervals where $\Phi(s_1) \leq 0, \Phi(s_2) \geq 0$. The separating set in the plane (m, ℓ) coincides with the discriminant set of the polynomial $(a^2 - s^2)(b^2 - s^2)\Phi(s)$ (a system of straight lines) and a half-line $\{\ell = 0, m < 0\}$ (the latter follows directly from the definition of the function L).

Consider the following polynomial in two auxiliary variables:

$$\Psi(s_1, s_2) = 4ms_1s_2 - 2\ell(s_1 + s_2) + (\ell^2 - 1)/m.$$

For variables (2), we obtain explicit dependencies on s_1, s_2 :

$$\begin{aligned} x_1 &= -\frac{r^2}{2(s_1 - s_2)^2}[\Psi(s_1, s_2) + \sqrt{\Phi(s_1)\Phi(s_2)}], \\ x_2 &= -\frac{r^2}{2(s_1 - s_2)^2}[\Psi(s_1, s_2) - \sqrt{\Phi(s_1)\Phi(s_2)}], \\ y_1 &= 2\frac{(2s_1s_2 - p^2) - 2\sqrt{(s_1^2 - a^2)(s_2^2 - b^2)}}{\Psi(s_1, s_2) - \sqrt{\Phi(s_1)\Phi(s_2)}}, \\ y_2 &= 2\frac{(2s_1s_2 - p^2) + 2\sqrt{(s_1^2 - a^2)(s_2^2 - b^2)}}{\Psi(s_1, s_2) + \sqrt{\Phi(s_1)\Phi(s_2)}}, \\ z_1 &= \frac{r}{s_1 - s_2}(\sqrt{s_1^2 - a^2} + \sqrt{s_2^2 - b^2}), \\ z_2 &= \frac{r}{s_1 - s_2}(\sqrt{s_1^2 - a^2} - \sqrt{s_2^2 - b^2}), \\ w_1 &= r\frac{\sqrt{\Phi(s_2)} - \sqrt{\Phi(s_1)}}{\Psi(s_1, s_2) - \sqrt{\Phi(s_1)\Phi(s_2)}}, \\ w_2 &= r\frac{\sqrt{\Phi(s_2)} + \sqrt{\Phi(s_1)}}{\Psi(s_1, s_2) + \sqrt{\Phi(s_1)\Phi(s_2)}}, \\ w_3 &= \frac{1}{s_1 - s_2}[\sqrt{(s_2^2 - b^2)\Phi(s_1)} - \sqrt{(s_1^2 - a^2)\Phi(s_2)}]. \end{aligned}$$

From here we immediately obtain explicit expressions for the phase variables α_i, β_j , and ω_k ($i, j, k = 1, 2, 3$) in terms of the separated variables.

Thus, we have completed the solution of the problem of the motion of a generalized Kovalevskaya top on the invariant submanifold N .

R E F E R E N C E S

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